

H-HARMONIC MAASS-JACOBI FORMS: WHY ZWEGERS'S μ -FUNCTION SPLITS

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ABSTRACT. The μ -function defined by Zwegers in order to study mock theta functions splits into the sum of a meromorphic and a one-variable real-analytic Jacobi form. We show that this splitting is a general phenomena, common to a large class of real-analytic Jacobi forms. The proof displays a novel kind of harmonic Maaß-Jacobi forms for higher rank indices, for which we develop a basic structure theory. Specializations of such Maaß-Jacobi forms to torsion points are related to mixed mock modular forms.

1. INTRODUCTION

In his celebrated thesis [Zwe02], Zwegers employed the so-called μ -function to provide an automorphic completion of the until then mysterious mock theta functions. The μ -function is a real-analytic Jacobi form of one modular and two elliptic variables. A remarkable fact was commented on by Zagier in [Zag07]: The “two-variable” μ -function can be written as the sum of a meromorphic Jacobi form and a real-analytic Jacobi form that only depends on the difference of the two elliptic variables.

$$\widehat{\mu}(\tau, u, v) = \frac{\zeta(\tau, u) - \zeta(\tau, v) + \zeta(\tau, u - v)}{\theta(\tau, u - v)} + \widehat{\mu}(\tau, u - v),$$

where ζ is the Weierstrass ζ -function, and θ is the Jacobi θ -function. The second term is the “one-variable” μ -function, which we denote, by abuse of notation, by the same letter as the original μ -function. The primary goal of this paper is to provide a theoretical framework which explains this curious splitting.

Classical Jacobi forms were defined in [EZ85], and have been applied in many contexts since then. In some cases, the generating functions of interesting arithmetic quantities are Jacobi forms [Zag91, GZ98]; in other cases, classical Jacobi forms and their generalizations have been used to understand the structure of modular forms of different type. For example, Jacobi forms occur as Fourier-Jacobi coefficient of holomorphic and non-holomorphic Siegel modular forms—see [Koh94, BRR11] for an explanation of how Fourier-Jacobi coefficients can be obtained from the latter. Jacobi forms also serve as a tool to better understand elliptic modular forms, quasimodular forms [Zag94], and mock modular forms. Quasimodular forms occur as Taylor coefficients of classical Jacobi forms. A more recent accomplishment that is based on Jacobi forms, and which is closely connected to the subject of this paper, are the findings by Zwegers in [Zwe02]. He provided three different ways, all based on non-classical Jacobi forms, to understand mock modular forms—see [Ram00] for details on the latter. First, he defined the μ -function, a real-analytic Jacobi form which specializes at certain torsion points to automorphic completions of mock theta functions. Second, he defined indefinite theta series for lattices of signature $(r - 1, 1)$, which are also real-analytic Jacobi forms. They can be employed in a similar way as the μ -function to understand mock theta functions. Third, Zwegers analyzed Fourier coefficients of meromorphic Jacobi forms of several elliptic variables, in order to obtain mock modular forms.

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Motivated by this success of real-analytic Jacobi forms (defined in an ad-hoc way), several attempts were made to give a precise definition of real-analytic Jacobi forms and, more specifically, harmonic Maaß-Jacobi forms. In the past few years, several such definitions, all based on the Casimir operator for the extended real Jacobi group, were suggested by Berndt and Schmidt, Pitale, Bringmann and Richter, Conley and the author, and Bringmann, Richter and the author [BS98, Pit09, BR10, CR11, BRR11]. In order to discuss these definitions, recall that Jacobi forms are functions $\phi : \mathbb{H} \times \mathbb{C}^N \rightarrow \mathbb{C}$, depending on a modular variable $\tau \in \mathbb{H} \subset \mathbb{C}$ in the Poincaré upper half plane and elliptic variables $z \in \mathbb{C}^N$. The index of a Jacobi form is a $N \times N$ matrix. A Jacobi form is semi-holomorphic, if it is holomorphic as a function of z . The Casimir operator is a certain invariant, central differential operator that annihilates constant functions.

Berndt and Schmidt, and Pitale gave definitions of real-analytic Jacobi forms that were motivated by representation theoretic ideas, thus restricting to functions that satisfy a polynomial growth condition with respect to the modular variable. By their definition, a real-analytic Jacobi form is an eigenfunction of the Casimir operator. In addition, Berndt and Schmidt require a real-analytic Jacobi to be an eigenfunction of another differential operator which is invariant, but *not* central, and which is similar to Δ^H in [BRR11]—see also Section 2. Pitale then showed that it suffices to consider semi-holomorphic forms in order to study smooth vectors in automorphic representations for the extended Jacobi group. This let him require Maaß-Jacobi forms to be semi-holomorphic eigenfunctions of the Casimir operator.

The work by Bringmann and Richter introduced a new idea. Restricting to functions that are annihilated by the Casimir operator, they relaxed the growth condition, requiring at most exponential growth, and called the Jacobi forms that arise this way harmonic Maaß-Jacobi forms. In order to distinguish them from the Jacobi forms mentioned so far, we will call them *harmonic weak Maaß-Jacobi forms*. It is important to notice that functions that only satisfy a weak growth condition currently cannot be incorporated into a satisfactory representation theoretic framework. However, they gain importance by the tremendous amount of applications in which harmonic weak Maaß forms [BF04] and harmonic weak Maaß-Jacobi forms show up—see, for example, [Bru02, DIT09, DMZ11]. Note that, even though Bringmann and Richter formally did not impose any further condition, their work treats only the semi-holomorphic case.

The weak growth condition was used in [CR11] to give a definition of *semi-holomorphic harmonic Maaß-Jacobi forms of lattice index*. In the context of [CR11], this type of Jacobi forms is relevant because of its connection with Siegel modular forms of higher genus [Rau12]. In Section 3 of [CR11], it was shown that the vanishing conditions with respect to analogs of the above Δ^H , in this setting, lead to semi-holomorphic functions, if the Jacobi index is not a scalar. This is an important observation, which is treated in more detail in Theorem 3.5 of this paper.

The case of scalar Jacobi indices was in the focus of [BRR12]. When allowing certain kinds of singularities, the class of harmonic Maaß-Jacobi forms that are annihilated by Δ^H is strictly larger than the class of semi-holomorphic harmonic Maaß-Jacobi forms. A complete structure theory of *Heisenberg-harmonic* (H-harmonic) *Maaß-Jacobi forms with scalar Jacobi indices* could be built up. We restrict ourselves to reminding the reader that the one-variable μ -function is an H-harmonic Maaß-Jacobi form.

The definition of H-harmonic Maaß-Jacobi forms. As discussed above, results on real-analytic Jacobi forms, so far, either restrict to the semi-holomorphic case or to the case of scalar Jacobi indices. The two-variable μ -function is neither semi-holomorphic and nor has it scalar Jacobi index. In order to study it as a real-analytic Jacobi form, we develop the theory of H-harmonic Maaß-Jacobi forms of arbitrary indices. To explain this definition, we

recall in detail the main result of Section 3 in [CR11]. Let Y_{\pm} denote the lowering and raising operators with respect to the elliptic variables—explicit expressions are given in Section 2. These are N -vectors of differential operators, and we write $Y_{\pm,e}$ for the linear combination of their entries with coefficients given by $e \in \mathbb{R}^N \setminus \{0\}$. We call them the lowering and raising operator in direction of e . In [CR11], it is shown that, if $N > 1$, any smooth function from $\mathbb{H} \times \mathbb{C}^N$ to \mathbb{C} that is annihilated by $Y_{+,e} \circ Y_{-,e}$ for all $e \in \mathbb{R}^N \setminus \{0\}$ is holomorphic in z . In other words, a coordinate independent definition of H-harmonic Maaß-Jacobi forms leads to semi-holomorphic forms whenever the Jacobi index is not scalar. From the perspective taken in [CR11], coordinate independence is a reasonable assumption, which holds automatically for Maaß-Jacobi forms that are obtained from real-analytic Siegel modular forms and orthogonal modular forms. The two-variable μ -function, however, is not semi-holomorphic. One is thus led to consider coordinate dependent H-harmonicity. Fixing a tuple E of linear independent vectors in \mathbb{Q}^N , we consider *Heisenberg harmonic functions with respect to E* ($H[E]$ -harmonic functions) that, by definition, are annihilated by $Y_{+,e} \circ Y_{-,e}$ for all $e \in E$ —see Definition 3.1. The basic theory of H-harmonic Maaß-Jacobi forms is developed in Section 3.

Splittings of H-harmonic Maaß-Jacobi forms. The μ -function falls under Definition 3.1, and its splitting can be explained in the setting of H-harmonic Maaß-Jacobi forms. We prove the following theorem, which shows that the splitting into a meromorphic Jacobi form and an H-harmonic Maaß-Jacobi forms that depends only on one elliptic variable is a general phenomenon.

Theorem 1.1. *Let ϕ be an $H[E]$ -harmonic Maaß-Jacobi form (defined in 3.1) of index L that is degenerate and has signature $(0, 1)$. Suppose that $E = (e)$ for $e \in \mathbb{Q}^{\text{rk}(L)}$ with length $L[e] < 0$. Further, suppose that*

- (1) *ϕ is annihilated by the operator ξ , which is defined in (2.6);*
- (2) *the singularities of ϕ lie in the union of sets defined by equations $\lambda(z) = \alpha\tau + \beta$, where $\lambda \in (\mathbb{R}^{\text{rk}(L)})^{\sim}$ is a linear form satisfying $\lambda(e) \neq 0$, and $\alpha, \beta \in \mathbb{Q}$;*
- (3) *ϕ satisfies the growth condition $\phi(\tau, \alpha\tau + \beta) = O(1)$ as $y \rightarrow \infty$ for all $\alpha, \beta \in \mathbb{Q}^{\text{rk}(L)}$ such that $\tau \mapsto (\tau, \alpha\tau + \beta)$ does not meet any singularities of ϕ .*

Then

$$(1.1) \quad \phi(\tau, z) = \phi^{\text{ra}}(\tau, \langle z, e \rangle_{\text{eucl}}) + \phi^{\text{mero}}(\tau, z).$$

Here, ϕ^{ra} is an H-harmonic Maaß-Jacobi form in the sense of [BRR12], ϕ^{mero} is meromorphic Jacobi form, and $\langle \cdot, \cdot \rangle_{\text{eucl}}$ denotes the euclidean scalar product.

In analogy with the results obtained in [BRR12], the first conditions on ϕ is most likely not necessary. A detailed study of H-harmonic Maaß-Jacobi forms that do not vanish under ξ , however, is not within the scope of this paper.

The following examples illustrate that the second and third condition are necessary. Recall the function $\phi_{-2,0}$ which was defined in [EZ85]. It is a weak Jacobi form of weight -2 and index 0 . We find that $\phi_{-2,0}(\tau, z_1)^{-1} \widehat{\mu}(\tau, z_2)$ is an H-harmonic Maaß-Jacobi form of weight $\frac{5}{2}$ and degenerate index $\begin{pmatrix} 0 & 0 \\ 0 & -1/2 \end{pmatrix}$, which does not satisfy the second condition. Using Proposition 3.6, it is easy to see from its Fourier expansion that it cannot be written as in (1.1). The same argument shows that $\phi_{-2,0}(\tau, z_1) \widehat{\mu}(\tau, z_2)$, which does not satisfy the third condition, can also not be written as in (1.1).

Constructions of H-harmonic Maaß-Jacobi forms. It is natural to ask for further examples of H-harmonic Maaß-Jacobi forms. We give two constructions. One of them generalizes theta series for indefinite lattices defined in [Zwe02]. The other is based on a theta-like decomposition as in [BRR12]. We prove modularity of indefinite theta series for non-degenerate

indefinite lattices in Section 4. Our definition of indefinite theta series follows closely the one suggested in [Zwe02]. It makes use of additional data which can be interpreted as a partial frame of the underlying lattice. This frame must satisfy strong conditions so that the associated indefinite theta series converges. In conjunction with the results in Section 3, however, one finds that this is no actual restriction. Theorem 3.5 shows that indefinite theta series as in Definition 4.3 can essentially only converge for “partial orthogonal frames” (which are described in the preliminaries).

The theta-like decomposition introduced for Maaß-Jacobi forms of scalar Jacobi index in [BRR12] provides a more flexible way to construct examples. Section 5 contains a consideration of the general case. We prove Theorem 5.2, which extends the theta-like decomposition studied so far to the case of arbitrary Jacobi indices. Recall the statement in the case of scalar indices. An H -harmonic Maaß-Jacobi form ϕ of index $-m < 0$ can be written as

$$\sum_{l \pmod{2m}} h_l(\tau) \widehat{\mu}_{m,l}(\tau, z) + \psi,$$

where the h_l are the components of a vector-valued elliptic modular form, the $\widehat{\mu}_{m,l}$ are functions depending only on m and l , and ψ is a meromorphic Jacobi form. It is important to note that in our setting, such a decomposition result must incorporate additional meromorphic factors. A prototypical decomposition for a Jacobi form of index $\begin{pmatrix} 1 & 0 \\ 0 & -m \end{pmatrix}$ is

$$\sum_{l \pmod{2m}} \psi_{m,l}(\tau, z_1) h_l(\tau) \widehat{\mu}_{m,l}(\tau, z) + \psi,$$

where the $\psi_{m,l}$ are meromorphic Jacobi forms, that depend on ϕ .

Specializations of H -harmonic Maaß-Jacobi forms. In the spirit of, for example, [Zwe02], it is interesting to study specializations of H -harmonic Maaß-Jacobi forms to torsion points. In Section 5, we describe the form of such specializations. Let $M_k^!$ and \widehat{M}_k denote, respectively, the space of weakly holomorphic modular forms and harmonic weak Maaß forms of weight k .

Theorem 1.2. *Let ϕ be an $H[E]$ -harmonic Maaß-Jacobi form for a rational frame E . Suppose that ϕ vanishes under ξ , and that the Jacobi index L of ϕ is non-degenerate. If $z : \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{C}^{\text{rk}(L)}$ is a torsion point that does not meet any singularities of ϕ , then*

$$\phi(\tau, z(\tau)) \in \bigoplus_{n=0}^{\#E} (\widehat{M}_{\frac{1}{2}})^{\otimes n} \otimes M_{k-\frac{n}{2}}^!.$$

By specializing H -harmonic Maaß-Jacobi forms to torsion points one obtains sums of products of harmonic weak Maaß forms. Such products cannot be characterized by differential operators, a paucity which let emerge the approach of *mixed mock modular forms* [Zag09]. Our results reconcile the approach taken by geometers, who tend to focus on harmonic modular forms, and physicists, who often encounter products of harmonic weak Maaß forms and holomorphic modular forms. In particular, physicists will want to study which of the mixed modular forms they have encountered so far can be obtained as “holomorphic parts” of specializations of H -harmonic Maaß-Jacobi forms to torsion points. A candidate for a mixed mock modular form that arises from this way can be found in [Man10, Equation (3.5)]. It already carries the structure possibly inherited by a theta-like decomposition of H -harmonic Maaß-Jacobi forms. In general, such Maaß-Jacobi forms would encode more detailed information than the mixed mock modular forms that are obtained from them.

2. PRELIMINARIES

2.1. Jacobi forms. The Poincaré upper half plane is

$$\mathbb{H} = \mathbb{H}_1 := \{\tau = x + iy : y > 0\} \subset \mathbb{C}.$$

The natural action of $\mathrm{SL}_2(\mathbb{R})$ on \mathbb{H} is given by

$$\gamma\tau = \frac{a\tau + b}{c\tau + d},$$

where a typical element γ of $\mathrm{SL}_2(\mathbb{R})$ is written $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For any integer $N \geq 1$, we define the *Jacobi upper half plane* by $\mathbb{H}_{1,N} := \mathbb{H} \times \mathbb{C}^N$. Its imaginary part is $\mathfrak{Im} \mathbb{H}_{1,N} := \{y \in \mathbb{R} : y > 0\} \times \mathbb{R}$.

Set $e(x) = \exp(2\pi ix)$ for all $x \in \mathbb{C}$. We will write q for $e(\tau)$ and ζ for $e(\sum z_i)$. Given $r \in \mathbb{R}^N$, define ζ^r as $e(\sum z_i r_i)$.

Multiplication in the real Jacobi group

$$G_N^J := \mathrm{SL}_2(\mathbb{R}) \ltimes (\mathbb{R}^N \otimes \mathbb{R}^2)$$

is given by

$$(\gamma, \lambda, \mu) \cdot (\gamma', \lambda', \mu') = (\gamma\gamma', (\lambda, \mu)\gamma' + (\lambda', \mu')).$$

Here, a typical element of G_N^J is denoted by (γ, λ, μ) , where λ and μ are N -vectors. They correspond to an element in $\mathbb{R}^N \times \mathbb{R}^N$ under the natural identification with the set of $N \times 2$ -matrices $M_{N,2}(\mathbb{R})$. The action of γ' on $M_{N,2}(\mathbb{R})$ is the usual multiplication.

The real Jacobi groups acts on the Jacobi upper half plane via

$$(\gamma, \lambda, \mu)(\tau, z) = \left(\gamma\tau, \frac{z + \lambda\tau + \mu}{c\tau + d}\right).$$

Fix an integer k and an integral symmetric matrix $L \in M_N^T(\frac{1}{2}\mathbb{Z})$. By integral we mean that the diagonal elements of L lie in \mathbb{Z} . The discrete subgroup $\Gamma_N^J = \mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^N \otimes \mathbb{Z}^2) \subset G_N^J$, the *full Jacobi group*, acts on complex valued functions ϕ on $\mathbb{H}_{1,N}$. We define

$$\begin{aligned} &(\phi|_{k,L}(\gamma, \lambda, \mu))(\tau, \mu) \\ &= (c\tau + d)^{-k} e((-cL[z])(c\tau + d)^{-1} + 2\lambda Lz + L[\tau]) \phi((\gamma, \lambda, \mu)(\tau, \mu)). \end{aligned}$$

This is the usual *Jacobi slash action*. The *skew Jacobi slash action*, $|_{k,L}^{\mathrm{sk}}$, is built up similarly. The factor $(c\tau + d)^{-k}$ is replaced by $|c\tau + d|^{-1} (c\bar{\tau} + d)^{1-k}$.

We say that a function $\phi : \mathbb{H}_{1,N} \rightarrow \mathbb{C}$ has *non-moving singularities*, if there are finitely many linear maps $p : \mathbb{R}^N \rightarrow \mathbb{R}$ and finitely many $\alpha, \beta \in \mathbb{R}$, such that the singularities of ϕ are located at $\{z : p(z) = \alpha + \beta\tau\} + \mathbb{Z}^N + \tau\mathbb{Z}^N$. We say that such a singularity in (τ_0, z_0) has

- (i) meromorphic type,
- (ii) almost meromorphic type, or
- (iii) real-analytic quotient type,

if there is a neighborhood $U \subset \mathbb{H}$ of τ_0 , a function $z_0 : \mathbb{H} \rightarrow \mathbb{C}^N$ with $z_0(\tau_0) = z_0$, and a function $\psi : U \setminus \{(\tau, z_0(\tau)) : \tau \in U\}$ which

- (i) is meromorphic,
- (ii) is the quotient of a real-analytic by a holomorphic functions, or
- (iii) is the quotient of two real-analytic functions

such that $\phi - \psi$ can be continued real-analytically to a neighborhood of (τ_0, z_0) .

A complex valued function on $\mathbb{H}_{1,N}$ that is meromorphic as a function of the second component, z , will be called *semi-meromorphic*. Such a function is called semi-holomorphic if it has no singularities.

2.2. Lattices and frames. A real symmetric matrix $L \in M_N^T(\mathbb{R})$ can be identified with the Gram matrix of a quadratic form on \mathbb{R}^N , which we denote by $L[\cdot]$. The bilinear form attached to $L[\cdot]$ will be denoted by $\langle z, z' \rangle_L = L[z + z'] - L[z] - L[z']$. It extends to a sesquilinear form on $\mathbb{C}^N = \mathbb{R}^N \otimes \mathbb{C}$. We will also denote this extension by $\langle \cdot, \cdot \rangle_L$. The \mathbb{Z} -module \mathbb{Z}^N is an even lattice in \mathbb{R}^N , and we will freely identify L with this lattice. We write $\text{rk}(L)$ for the rank of L as a \mathbb{Z} -module. We write $|L|$ for the determinant of L . We will often assume that L is non-degenerate. That is, we will assume that the matrix L is invertible. A vector $\nu \in L$ is called isotropic if $L[\nu] = 0$. The maximal totally isotropic subspace $\{l \in L : \forall l' \in L : \langle l, l' \rangle_L = 0\}$ will be denoted by L_0 . Write dV_+ for the number of positive eigenvalues of L over \mathbb{R} . Write dV_- for the number of negative eigenvalues. Then the signature of the lattice L is the pair (dV_+, dV_-) .

We will call L positive or negative semi-definite if $dV_- = 0$ or $dV_+ = 0$. We say that L is positive or negative definite if $dV_+ = \text{rk}(L)$ or $dV_- = \text{rk}(L)$. A totally isotropic lattice satisfies $dV_+ = dV_- = 0$ by definition.

The dual of a non-degenerate lattice L will be denoted by L^\vee . We will write $\text{disc}(L)$ for the discriminant module L^\vee / L . The Weyl representation associated to L is a representation of the metaplectic cover of $\text{SL}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[\text{disc}(L)]$ (see [Sko07]). A basis for $\mathbb{C}[\text{disc}(L)]$ is given in terms of b_λ where λ runs through $\text{disc}(L)$. By abuse of notation we write J and T for the generators of the metaplectic cover of $\text{SL}_2(\mathbb{Z})$ that project to the corresponding generators $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ of $\text{SL}_2(\mathbb{Z})$. In terms of said basis and these generators, the Weyl representation is defined as:

$$\begin{aligned} \rho_L(T) b_\lambda &:= e(L[\lambda]) b_\lambda, \\ \rho_m(S) b_\lambda &:= \frac{1}{\sqrt{2i|L|}} \sum_{\lambda' \in \text{disc}(L)} e(-\langle \lambda, \lambda' \rangle_L) b_{\lambda'}. \end{aligned}$$

Throughout the paper we will freely pass to the metaplectic cover of $\text{SL}_2(\mathbb{R})$ whenever necessary.

Given any set of vectors ν_1, \dots, ν_n we denote their span by $\text{span}(\nu_1, \dots, \nu_n)$. It will be clear by context, whether we mean the span over \mathbb{Z} , \mathbb{R} , or \mathbb{C} . The orthogonal complement of ν_1, \dots, ν_n will be denoted by $\{\nu_1, \dots, \nu_n\}^\perp$.

A frame E of \mathbb{R}^N is an (ordered) basis. We write $e \in E$ if e is an element of the underlying unordered basis. Given two frames $E', E'' \subseteq E$, we mean by $E' \cup E''$ the frame whose elements occur in the same order as in E . The set of vectors in E which have positive or negative length or are isotropic will be denoted by E_+ , E_- , or E_0 , respectively. A partial frame is a basis for a subspace of \mathbb{R}^N . We say that a partial frame is orthogonal if its elements are orthogonal to each other with respect to the euclidean scalar product on \mathbb{R}^N . By abuse of notation we will often say that E is a frame of L , when we mean it is a frame of $\mathbb{R}^N = L \otimes \mathbb{R}$. Throughout the paper, we assume that the elements of E have euclidean length 1.

There are natural coordinates of \mathbb{C}^N attached to a frame E of L . We will write $v_e = \langle v, e \rangle_{\text{eucl}}$ for any $e \in E$. Here, $\langle \cdot, \cdot \rangle_{\text{eucl}}$ means the Euclidean scalar product. Analogously, ζ_e means $e(z_e)$. By abuse of notation we also write v_e for $v_e e / |e|_{\text{eucl}}$ and v_E for $\sum_{e \in E} v_e e / |e|_{\text{eucl}}$. Since one quantity is a number and the other a vector, it will be clear from the context, which one we refer to. We write ∂_{z_e} for the derivative with respect to z_e of a function depending on $(z_{e'})_{e' \in E}$.

We will often write L_e for $L[e]$. The restriction of L to $\text{span}(E)$ will be denoted by L_E . In particular, L_e can denote a number or a lattice, depending on the context.

2.3. Differential operators. The following lowering and raising operators have been given in [CR11] in the case of non-degenerate L . The fact that they generate the algebra of covariant

operators of the extended real Jacobi group is proved there. We content ourselves with giving explicit expressions. The reader is referred to [CR11] for a detailed treatment.

In order to treat the degenerate case, define π_{nd} as the projection onto L_0^\perp , where in this case we take the orthogonal complement with respect to the euclidean scalar product. Writing $L = 2\pi i L$, the *lowering operators*, X_- and Y_- , and the *raising operators*, X_+ and Y_+ , are

$$\begin{aligned} X_-^{k,L} &:= -2iy(y\partial_{\bar{\tau}} + (\partial_{\text{nd}}v)^T\partial_{\bar{z}}), & X_+^{k,L} &:= 2i(\partial_{\tau} + y^{-1}v^T\partial_z + y^{-2}L[\pi_{\text{nd}}v]) + ky^{-1}, \\ Y_-^{k,L} &:= -iy\partial_{\bar{z}}, & Y_+^{k,L} &:= i\partial_z + 2iy^{-1}L\pi_{\text{nd}}v. \end{aligned}$$

The *Casimir operator* commutes with all other covariant differential operators. In order to define it, we set

$$(2.1) \quad \Delta_k := 4y^2\partial_{\tau}\partial_{\bar{\tau}} - 2kiy\partial_{\bar{\tau}}.$$

An explicit expression for the Casimir operator for it was found in [CR11], again restricting to the non-degenerate case. Let L_{nd}^{-1} be the matrix with $\ker L_{\text{nd}}^{-1} = \ker \pi_{\text{nd}}$ and $L_{\text{nd}}L_{\text{nd}}^{-1} = L_{\text{nd}}^{-1}L_{\text{nd}} = \pi_{\text{nd}}$. Then the Casimir operator is given by

$$\begin{aligned} (2.2) \quad \mathcal{C}_{k,L}^J &:= -2\Delta_{k-N/2} + 2y^2(\partial_{\bar{\tau}}L_{\text{nd}}^{-1}[\partial_z] + \partial_{\tau}L_{\text{nd}}^{-1}[\partial_{\bar{z}}]) - 8y\partial_{\tau}(\pi_{\text{nd}}v)^T\partial_{\bar{z}} \\ &\quad - \frac{1}{2}y^2(L_{\text{nd}}^{-1}[\partial_{\bar{z}}]L_{\text{nd}}^{-1}[\partial_z] - (\partial_{\bar{z}}^T L_{\text{nd}}^{-1}\partial_z)^2) + 2y((\pi_{\text{nd}}v)^T\partial_{\bar{z}})\partial_z^T L_{\text{nd}}^{-1}\partial_u \\ &\quad - \frac{1}{2}(2k - N + 1)iy\partial_{\bar{z}}^T L_{\text{nd}}^{-1}\partial_u + 2(\pi_{\text{nd}}v)^T((\pi_{\text{nd}}v)^T\partial_{\bar{z}})\partial_{\bar{z}} \\ &\quad + (2k - N - 1)i(\pi_{\text{nd}}v)^T\partial_{\bar{z}}. \end{aligned}$$

The Heisenberg Laplace operator will be equally important in this paper. Given a frame E of \mathbb{R}^N and $e \in E$, define

$$(2.3) \quad \Delta_L^{\text{H}[e]} := y\partial_{z_e}\partial_{\bar{z}_e} + (\pi_{\text{nd}}v)^T L e \partial_{\bar{z}_e}.$$

The heat operator is defined as follows:

$$(2.4) \quad \mathbb{L}_L := 2\partial_{\tau} - \frac{1}{2}L_{\text{nd}}^{-1}[\partial_z].$$

We also define a partial heat operator

$$(2.5) \quad \mathbb{L}_E := 2\partial_{\tau} - \frac{1}{2}L_E^{-1}[\partial_z],$$

if L_E is non-degenerate.

Following the ideas in [BR10] and [CR11], we build a ξ -operator out of the covariant operators. It can be used to relate various spaces of real-analytic Jacobi forms. Note, however, that we apply complex conjugation, which is not applied in neither of the references. Define

$$(2.6) \quad \xi_{k,L}(\phi) := \overline{y^{k-2-N/2}(X_- - \frac{i}{2}L_{\text{nd}}^{-1}[Y_-])(\phi)}.$$

For a partial frame E such that L_E is non-degenerate, define

$$(2.7) \quad \xi_{k,L}^E(\phi) := \overline{y^{k-2-N/2}(X_- - \frac{i}{2}L_E^{-1}[Y_-])(\phi)}.$$

It is, of course, possible to define ξ^E without the condition on L_E , but we avoid this technicality as we will not need such ξ^E .

2.4. Elliptic modular forms. Vector-valued elliptic modular forms are invariant under the $|_{k,\rho}$ -action associated to some $k \in \frac{1}{2}\mathbb{Z}$ a finite dimensional, complex representation $\rho : \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{GL}(V)$. It is defined as

$$(f|_{k,\rho}\gamma)(\tau) = (c\tau + d)^{-k} \rho(\gamma) f(\gamma\tau),$$

where $\gamma\tau$ denotes the action of $\mathrm{SL}_2(\mathbb{R})$ on \mathbb{H} . Recall that we freely pass to the metaplectic cover in case $k \notin \mathbb{Z}$. The space of vector-valued modular forms of weight k and type ρ is the space of functions $f : \mathbb{H} \rightarrow V$ satisfying

- (1) $f|_k \gamma = f$ for all $\gamma \in \Gamma$,
- (2) $f(\tau) < O(1)$ as $y \rightarrow \infty$.

This is denoted by $M_k(\rho)$.

The space of weakly holomorphic elliptic modular forms is the space of functions $f : \mathbb{H} \rightarrow V$ where the second condition is weakend: $f(\tau) < O(e^{ay})$ for some $a > 0$ as $y \rightarrow \infty$. This space is denoted by $M_k^!(\rho)$.

The space of harmonic weak Maaß forms of weight k is denoted by \widehat{M}_k . It is the space of real-analytic functions $f : \mathbb{H} \rightarrow \mathbb{C}$ satisfying

- (1) $f|_k \gamma = f$ for all $\gamma \in \Gamma$,
- (2) $\Delta_k f = 0$,
- (3) $f(\tau) < O(e^{ay})$ as $y \rightarrow \infty$ for some $a > 0$.

2.5. Skew Jacobi forms. We end this section with the definition of skew-holomorphic Jacobi forms. In the case of scalar Jacobi indices it was given in [Sko90]. Originally, skew-holomorphic functions of weight k are invariant under $|\frac{1}{2}, k - \frac{1}{2}$. The reader should be alerted to the fact that we use the conjugate $|_{k - \frac{1}{2}, \frac{1}{2}}$, which is more convenient in the context of the Heisenberg ξ -operators to be defined later. For $\alpha, \beta \in \mathbb{R}$ with $\alpha - \beta \in \frac{1}{2}\mathbb{Z}$, we define

$$(f|_{\alpha,\beta}\gamma)(\tau) = (c\tau + d)^{-\alpha} (c\bar{\tau} + d)^{-\beta} \rho(\gamma) f(\gamma\tau).$$

Note that the condition on $\alpha - \beta$ ensures that this is well-defined as an action of the metaplectic cover of $\mathrm{SL}_2(\mathbb{Z})$. This action naturally induces an action $|_{\alpha,\beta,L}$ of the full Jacobi group. We write $|_{k,L}^{\mathrm{sk}[E]}$ for $|_{k - \frac{\#E}{2}, \frac{\#E}{2}, L}$.

Definition 2.1. Let L be a non-degenerate lattice and E a partial frame. Let $\phi : \mathbb{H}_{1,N} \rightarrow \mathbb{C}$ be real-analytic except for non-moving singularities of real-analytic quotient type. We call ϕ an E -skew Jacobi form (with singularities) of weight $k \in \mathbb{Z}$ and index L if it satisfies the following conditions:

- (i) For all $\gamma^J \in \Gamma_{\mathrm{rk}(L)}^J$, we have $\phi|_{k,L}^{\mathrm{sk}[E]} \gamma^J = \phi$.
- (ii) ϕ is annihilated by $\xi_{k,L}^E$ and $\overline{\mathbb{L}}_E$.
- (iii) ϕ is holomorphic in z_{E^\perp} and anti-holomorphic in z_E .
- (iv) For some $a > 0$, $\phi(\tau, z) = O(e^{ay})$ as $y \rightarrow \infty$ if $z = \alpha\tau + \beta$, $\alpha, \beta \in \mathbb{R}^{\mathrm{rk}(L)}$ is a non-singular point.

We will denote the space of E -skew Jacobi forms of weight k and index L by $\mathcal{MJ}_{k,L}^{\mathrm{sk}[E]}$. As we allow singularities throughout the paper, we will suppress the phrase “with singularities”.

As in the holomorphic and skew-holomorphic case, we have a theta decomposition.

Proposition 2.2. Suppose that ϕ is an E -skew Jacobi form of weight k and index $L = L_E \oplus L_{E^\perp}$. If ϕ has singularities only at $\lambda_i(z) = \alpha_i\tau + \beta_i$, where $\lambda_i \in (E^\perp)^\vee$, $\alpha_i, \beta_i \in \mathbb{Q}$, then

$$\phi(\tau, z) = \sum_{r \in L_E} \phi_r(\tau, z_{E^\perp}) \overline{q^{L^{-1}[r]/2} \zeta^r},$$

for meromorphic ϕ_r , and $L_E > 0$.

3. H-HARMONIC MAASS-JACOBI FORMS

In [CR11], it was argued that all H-harmonic Maaß-Jacobi form for non-degenerated lattice indices of rank greater than 2 are semi-meromorphic. The proof of this fact depends on the independence of coordinates. We are led to assume this independence, because a lattice is not equipped with any basis. After fixing a basis, that is, when considering framed lattices, examples can be found that are not semi-meromorphic.

Definition 3.1 (H-harmonic Maaß-Jacobi forms for framed lattices). *Let L be a lattice, and E a partial frame for L . Let $\phi : \mathbb{H}_{1, \text{rk}(L)} \rightarrow \mathbb{C}$ be real-analytic except for non-moving almost meromorphic singularities. We say that ϕ is an $H[E]$ -harmonic Maaß-Jacobi form of weight k , index L if it satisfies the following conditions:*

- (1) *For all $\gamma^J \in \Gamma_{\text{rk}(L)}^J$, we have $\phi|_{k,L} \gamma^J = \phi$.*
- (2) *ϕ is annihilated by $\mathcal{C}_{k,L}^J$.*
- (3) *ϕ is annihilated by all $\Delta_L^{H[e]}$ for all $e \in E$, and ϕ is holomorphic in z_{E^\perp} .*
- (4) *$\phi(\tau, z) = O(e^{ay})$ ($a > 0$) as $y \rightarrow \infty$ if $z = \alpha\tau + \beta$, $\alpha, \beta \in \mathbb{R}^{\text{rk}(L)}$ is a non-singular point.*

We denote the space of all $H[E]$ -harmonic Maaß-Jacobi forms of fixed weight k and index L by $\mathcal{MJ}_{k,L}^{\Delta, H[E]}$. We say that ϕ is H-harmonic, if it is $H[E]$ -harmonic for some E .

Remarks 3.2.

- (a) *The definition extends to half-integral weights and indices, and characters of $\Gamma_{\text{rk}(L)}^J$. We do not treat these cases, as they lead to additional technical difficulties and do not yield further insight. Zweigers's μ -function, however, strictly speaking does not fall under Definition 3.1.*
- (b) *An analog definition can be made for the skew Jacobi slash action.*
- (c) *We can define H-harmonic Maaß-Jacobi forms for any $\Gamma = \Gamma_{\text{ell}} \ltimes (\mathbb{Z}^2 \otimes \mathbb{Z}^{\text{rk}(L)})$ where $\Gamma_{\text{ell}} \subset \text{SL}_2(\mathbb{Z})$ has finite index.*
- (d) *Definition 3.1 is compatible with the notion of vector-valued Jacobi forms. For any complex representation ρ of $\text{SL}_2(\mathbb{Z})$, we set $\phi|_{k,L,\rho}(\gamma, \lambda, \mu) = \rho(\gamma)\phi|_{k,L}(\gamma, \lambda, \mu)$. We say that a function $\phi : \mathbb{H}_{1, \text{rk}(L)} \rightarrow \mathbb{C}^{\dim \rho}$ is a Jacobi form of weight k , index (L, E) , and type ρ if every component satisfies Conditions (2) to (4) of Definition 3.1 and $\phi|_{k,L,\rho} \gamma^J = \phi$ for all $\gamma^J \in \Gamma_{\text{rk}(L)}^J$.*

3.1. Subspaces of $\mathcal{MJ}_{k,L}^{\Delta, H[E]}$. Based on the lowering operators given in Section 2 and in analogy with the ideas in [BRR12], we define several subspaces of $\mathcal{MJ}_{k,L}^{\Delta, H[E]}$. The space

$$\mathcal{MJ}_{k,L}^{\delta, H[E]} := \ker (X_-^{k,L} |_{\mathcal{MJ}_{k,L}^{\Delta, H[E]}})$$

is the most important one. Indeed, the theory developed in this paper is mostly concerned with forms in $\mathcal{MJ}_{k,L}^{\delta, H[E]}$. For this reason, we will usually suppress the superscript δ .

For $E' \subseteq E$, we set

$$\mathcal{MJ}_{k,L}^{\Delta, H[E']} := \bigcap_{e \in E \setminus E'} \ker (Y_{+,e}^{k,L} |_{\mathcal{MJ}_{k,L}^{\Delta, H[E]}}).$$

In accordance with the notation in [BRR12], we write $\mathcal{MJ}_{k,L}^{\Delta, h}$ for $\mathcal{MJ}_{k,L}^{\Delta, H[\emptyset]}$. Notice that in this case we can suppress the frame $E = \emptyset$, because the space of semi-meromorphic Maaß-Jacobi forms is independent of it.

Remarks 3.3.

- (a) The theory of Maaß-Jacobi forms in $\mathcal{MJ}_{k,L}^{\Delta,h}$ was developed in [CR11]. It resembles strongly the theory of harmonic weak Maaß forms.
- (b) A theory similar to the theory presented in [BRR12] could be developed for $\mathcal{MJ}^{\Delta,H[E]}$. A decomposition $\mathcal{MJ}_{k,L}^{\Delta,H[E]} = \sum_{E'} \mathcal{MJ}_{k,L}^{\delta,H[E']} \otimes \mathcal{MJ}_{k,L_{E'\perp}}^{\Delta,h}$, where the sum runs over (possibly trivial) partial frames E' such that $L = L_{E'} \oplus L_{E'\perp}$ probably holds. The author did not check any details.
- (c) Note that $\mathcal{MJ}_{k,L}^h$ is the space of meromorphic Jacobi forms with non-moving singularities of meromorphic type.

As in the case of $\text{rk}(L) = 1$, dealt with in [BRR12], there is a Heisenberg ξ -operator, which we will define now. Let E be a partial orthogonal frame for L . Suppose that L admits a decomposition $L_E \oplus L_{E^\perp}$ where L_E is negative definite. Define

$$\xi^{H[E]}(\phi) := y^{-\text{rk}(L_E)/2} \exp(-4\pi L_E[v_E]/y) \left(\prod_{e \in E} Y_{-,e}^{k,L} \right)(\phi).$$

We will not describe the image of $\xi^{H[E]}$ in the most general situation. It consists of “ E' -skew $H[E]$ -harmonic” Maaß-Jacobi form, which we will not use.

Remark 3.4. By Proposition 3.6, $H[E]$ -harmonic Maaß-Jacobi forms are semi-holomorphic in z_e , if $e \in E, L[e] > 0$. For this reason, if L is non-degenerate, the above Heisenberg ξ -operators are sufficient to simplify H -harmonic Maaß-Jacobi forms to semi-holomorphic ones—see Section 5. If L is not invertible, consider the statement of Theorem 1.1.

3.2. Orthogonality of E with respect to L . We connect H -harmonic Maaß-Jacobi forms for a framed lattice to coordinate independent H -harmonic Maaß-Jacobi forms, which have been studied in [CR11]. Recall that by the result in [CR11], any Maaß-Jacobi form that is H -harmonic for all frames of a non-degenerate lattice L is semi-holomorphic. The following theorem refines this statement, by relating it to L -orthogonality of E .

Theorem 3.5. Let L be a lattice, and E a partial frame. Set

$$E' = \{e \in E : \forall e' \in E, e' \neq e : \langle e, e' \rangle_L = 0\}.$$

Then

$$\mathcal{MJ}_{k,L}^{\Delta,H[E]} = \mathcal{MJ}_{k,L}^{\Delta,H[E']}.$$

Proof. We must prove that any $\phi \in \mathcal{MJ}_{k,L}^{\Delta,H[E]}$ depends holomorphically on z_e whenever there is $e' \in E, e' \neq e$ such that $\langle e, e' \rangle \neq 0$. The proof of [CR11, Theorem 3.4] can be applied word by word to this setting. \square

From now on, we will assume that all (partial) frames E satisfy

$$(3.1) \quad \forall e, e' \in E, e \neq e' : \langle e, e' \rangle_L = 0.$$

3.3. Fourier expansions. We will study the Fourier expansions of H -harmonic Maaß-Jacobi forms. This will yield further restrictions on the index of Maaß-Jacobi forms which are not semi-holomorphic.

Besides the holomorphic terms, two building blocks turn out to be sufficient to explicitly describe the Fourier expansion. The function H , up to slight modifications, appeared in [BF04] first. It is usually employed to describe Fourier expansions of harmonic weak Maaß forms. The

function $H^{\mathbf{H}[e]}$, adopted from [BRR12], was adjusted to the setting of lattices indices. Set

$$(3.2) \quad H(y; D) := e^{-y} \int_{-2y}^{\infty} e^{-t} t^{-k-N/2} dt \Big|_{y=\pi Dy/2|L|}, \quad \text{if } D \neq 0;$$

$$(3.3) \quad H(y; D) := y^{-k+\text{rk}(L)/2}, \quad \text{if } D = 0; \quad \text{and}$$

$$(3.4) \quad H^{\mathbf{H}[e]}(y, v; r) := \text{sgn}(r_e + 2L_e \frac{v_e}{y}) \gamma\left(\frac{1}{2}, \frac{-y\pi}{L_e} (r_e + 2L_e \frac{v_e}{y})^2\right).$$

Here, the incomplete gamma function $\int_0^y t^{s-1} e^t dy$ is denoted by $\gamma(s, t)$. In the last line, we assume that $L_e < 0$. If the integral representation for H does not converge, we mean its analytic continuation in k .

For $D < 0$, we have

$$H(y; D) = \exp(-\pi Dy/2|L|) \Gamma(1 + \frac{N}{2} - k, -Dy/|L|)$$

Recall that L_{nd} is a maximal non-degenerate sublattice of L . We say that

$$D_L(n, r) := |L_{\text{nd}}|(4n - L_{\text{nd}}^{-1}[r])$$

is the discriminant of an index (n, r) , $n \in \mathbb{Z}$, $r \in L$. We always suppress the dependence on L , n , and r , which will be clear by the context.

Maaß-Jacobi forms with non-moving singularities admit a local Fourier expansion (see [BRR12]). The next proposition describes the analytic part of Fourier terms that occur for H-harmonic Maaß-Jacobi forms.

Proposition 3.6. *Let ϕ be an $\mathbf{H}[E]$ -harmonic Maaß-Jacobi of index L . Write*

$$(3.5) \quad \sum_{n \in \mathbb{Z}, r \in \mathbb{Z}^N} c(n, r; y, v) q^n \zeta^r$$

for the local Fourier expansion of ϕ around some point in $\mathbb{H}_{1, \text{rk}(L)}$. For fixed n and r , $c(n, r)$ is the sum of complex multiples of

$$(3.6) \quad \sum_{E' \subseteq E_0 \cup E_-} c^{\delta, \mathbf{H}[E']}(n, r) \prod_{e \in E'_0} \exp(4\pi r_e v_e) \prod_{e \in E'_-} H^{\mathbf{H}[e]}(y, v; r),$$

and

$$(3.7) \quad \sum_{E' \subseteq E_0 \cup E_-} c^{\Delta, \mathbf{H}[E']}(n, r) H(y; D) \prod_{e \in E'_0} \exp(4\pi r_e v_e) \prod_{e \in E'_-} H^{\mathbf{H}[e]}(y, v; r).$$

We state two corollaries, before proving the proposition.

Corollary 3.7. *A Maaß-Jacobi form $\phi \in \mathcal{MJ}_{k,L}^{\Delta, \mathbf{H}[E']}$ lies in $\mathcal{MJ}_{k,L}^{\delta, \mathbf{H}[E']}$ if and only if all $c^{-[E'']}$ ($E'' \subseteq E'$) in the Fourier expansion (3.5) of ϕ vanish.*

Proof. Using the Fourier expansion given in Proposition 3.6, the corollary can be proved locally. Since Maaß-Jacobi forms are real-analytic with non-moving singularities, it holds everywhere, by analytic continuation. \square

Corollary 3.8. *Adopting E' from Theorem 3.5, we have*

$$\mathcal{MJ}_{k,L}^{\mathbf{H}[E]} = \mathcal{MJ}_{k,L}^{\mathbf{H}[E']}.$$

Proof. This follows from Theorem 3.5 and Corollary 3.7. \square

Proposition 3.6 will follow from the following two observations. It is important to keep Assumption (3.1) in mind.

Proposition 3.9. *Let L be a lattice and E a frame for L . Fix some $e \in E$, that is not isotropic. Then we have*

$$\Delta^{\mathbf{H}[e]} a(v_e^2/y) = \frac{\Delta_{\frac{1}{2}} a(y) e(L_e \tau)}{y e(L_e \tau)} \Big|_{y \rightarrow v_e^2/y}.$$

In particular, the right hand side extends smoothly to $v_e = 0$.

Proposition 3.10. *Let E be a frame of \mathbb{R}^N . Suppose that $f_e(\cdot; r) : \mathfrak{Im} \mathbb{H}_{1,N} \rightarrow \mathbb{C}$ are solutions to $\Delta^{\mathbf{H}[e]} f_e(y, v_e; r) \zeta^r = 0$ for all $e \in E$. Then*

$$\mathcal{C}_{k,L}^J a(y) q^n \prod_{e \in E} f_e(y, v_e; r) \zeta^r = 0$$

if and only if

$$\Delta_{k-N/2} a(y) q^{D/4|L_{\text{nd}}|} = 0.$$

The proof of Proposition 3.9 is straight forward, but the proof of Proposition 3.10 is a tedious and unenlightening computation. We defer both to the end of this section.

Proof of Proposition 3.6. We first show that for fixed n and r , the Fourier term of index (n, r) can be written as in the assumptions of Proposition 3.10. In case of $L[e] \neq 0$ this follows from Proposition 3.9. In the case $L[e] = 0$ the fundamental solutions to $\Delta^{\mathbf{H}[e]}$ are $\zeta_e^{r_e}$ and $\bar{\zeta}_e^{r_e}$, so that we can choose $f_e(y, v_e; r) = 1$ or $f_e(y, v_e; r) = \exp(4\pi r_e v_e)$.

Using Proposition 3.10, it follows that for fixed n and r a term in the Fourier expansion of ϕ is of the form

$$(3.8) \quad \sum_{E' \subseteq E} (c^{\delta, \mathbf{H}[E']}(n, r) + c^{\Delta, \mathbf{H}[E']}(n, r) H(y; D)) \cdot \prod_{e \in E'_+} \mathcal{E}(y, v_e; r) \prod_{e \in E'_0} e(4\pi v_e r_e) \prod_{e \in E'_-} H^{\mathbf{H}[e]}(y, v; r) q^n \zeta^r.$$

Here, \mathcal{E} is essential the error function:

$$\mathcal{E}(y, v_e; r) = \sqrt{y(r_e - 2L_e \frac{v_e}{y})^2} \int_0^1 \exp\left(\frac{2\pi y}{L_e} (r_e - 2L_e \frac{v_e}{y})^2 u^2\right) du.$$

In order to prove that no term involving \mathcal{E} can occur, we may assume that $L = L_r \oplus \bigoplus_{e \in E} L_e$. Indeed, since E is rational, we can find integral basis vectors $w_1, \dots, w_{\text{rk}(L)} \in (\mathbb{R}E \cup E^\perp) \cap L$ of $\mathbb{R}^{\text{rk}(L)}$. Let A be the matrix with columns $w_1, \dots, w_{\text{rk}(L)}$. Then $\phi(\tau, Az)$ is \mathbf{H} -harmonic for the standard basis of $\mathbb{R}^{\text{rk}(L)}$, and its Jacobi index is diagonal.

Let e be a vector of positive length in E , such that $\mathcal{E}(y, v_e; r)$ occurs as a factor in the Fourier expansion 3.8. We choose a maximal E' subject to the conditions $e \in E'$ and $|c^{\delta, \mathbf{H}[E']}(n, r)| + |c^{\Delta, \mathbf{H}[E']}(n, r)| \neq 0$ for some (n, r) . Applying $\xi^{\mathbf{H}[E']}$ we obtain an E' -skew Jacobi form. By Lemma 5.5, it has no singularities at any $\lambda(z) = \alpha\tau + \beta$ for $\lambda \in \text{span}(E)^\vee$, $\alpha, \beta \in \mathbb{Q}$. Hence it has a theta decomposition with respect to z_E . In particular, its Jacobi index at e must be positive. This implies $L_e < 0$, contradicting the assumption on L_e . This completes the proof of Proposition 3.6. \square

3.4. Degenerate Jacobi indices. We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We may assume that e is the first standard basis vector. By Theorem 3.5, we may further assume that $L = L_e \oplus (0)^{\oplus \text{rk}(L)-1}$, because in all other cases the theorem's assertion holds when choosing $\phi^{\text{ra}} = 0$.

Apply $\xi^{\mathbf{H}[e]}$ to ϕ in order to obtain an e -skew Jacobi form $\tilde{\phi}$. By assumption and Lemma 5.5, $\tilde{\phi}$ has no singularities. Using the theta decomposition, we can consider it as an e -skew Jacobi

form of scalar index. Because of the growth condition satisfied by ϕ , it only depends on τ and z_e . By the results in [BRR12], there is an H-harmonic Maaß-Jacobi form ϕ^{ra} which maps to $\tilde{\phi}$ under $\xi^{\text{H}[e]}$ and vanishes under ξ . We find that $\phi^{\text{mero}} := \phi - \phi^{\text{ra}}$ is semi-meromorphic. Since $\xi(\phi^{\text{mero}}) = 0$, it is meromorphic. \square

Example 3.11. *We illustrate how to apply Theorem 1.1 to Zwegers's μ -function. As noted in the Remarks following Definition 3.1, we have not treated neither the half integral index case nor the case of characters, but the reader can easily check that the arguments needed to prove the Theorem apply to this case as well.*

In [Zwe02], the variables u and v are complex. For the time being, we adopt this notation. The μ -function is defined as

$$\begin{aligned} \hat{\mu}(\tau, u, v) &= \frac{e^{\pi i u}}{\theta(\tau, v)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{\pi i (n^2 + n)\tau + 2\pi i n v}}{1 - e^{2\pi i n \tau + 2\pi i u}} \\ &\quad - \frac{i}{2\sqrt{\pi}} \sum_{n \in \mathbb{Z} + \frac{1}{2}} \left(-\text{sgn}(n) - H^{\text{H}[(-1 \ 1)^T]}(y, \Im(u - v); n) \right) (-1)^{n + \frac{1}{2}} e^{-\pi i n^2 \tau + 2\pi i n(u - v)}. \end{aligned}$$

The second term contributes no singularities, and the first term is meromorphic. We see that $\hat{\mu}$ satisfies the assumptions of Theorem 1.1 with $e = (-1 \ 1)^T$. It follows that $\hat{\mu}$ decomposes as a meromorphic Jacobi form and a real analytic Jacobi form that only depends on $u - v$. We remind the reader that an explicit decomposition was given in [Zag07]—see this paper's introduction.

3.5. Proofs of Proposition 3.9 and 3.10.

Proof of Proposition 3.9. Using the fact that e is orthogonal to all other elements of E , we find that $\Delta^{\text{H}[e]} = y \partial_{z_e} \partial_{\bar{z}_e} + 4\pi i L_e v_e \partial_{\bar{z}_e}$. A straight forward computation yields

$$\Delta^{\text{H}[e]} a(v_e^2 / y) = \frac{v_e^2}{y} a''(v_e^2 / y) + \left(\frac{1}{2} - \frac{4\pi L_e v_e^2}{y} \right) a'(v_e^2 / y).$$

On the other hand $\Delta_{\frac{1}{2}} = 4y^2 \partial_\tau \partial_{\bar{\tau}} - iy \partial_{\bar{\tau}}$. From this we get

$$\Delta_{\frac{1}{2}} a(y) e(L_e \tau) = \left(y^2 a''(y) + \left(\frac{y}{2} - 4\pi L_e y^2 \right) a'(y) \right) e(L_e \tau).$$

Combining both equations, the statement is obvious. \square

Proof of Proposition 3.10. Because of the invariance of $\mathcal{C}^{\mathcal{M}} J_{k,L}$ and Δ_e^{H} , we can assume that $r_e = 0$ if $L[e] \neq 0$. In particular, we can assume that $D / 4|L_{\text{nd}}| = n$.

We first deduce three relations that hold under the assumption that $L[e] \neq 0$ for all $e \in E$. Sums and products, if not indicated differently, run over elements in E . If $L[e] \neq 0$, we can use Proposition 3.9 to find

$$(3.9) \quad \partial_y \prod_e f_e(y, v_e) \zeta_{E_0}^{r_{E_0}} = \sum_e \frac{-v_e}{2y} \partial_{v_e} \prod_{\tilde{e}} f_{\tilde{e}}(y, v_{\tilde{e}}) \zeta_{E_0}^{r_{E_0}}.$$

By applying this relation twice, we obtain

$$\begin{aligned} (3.10) \quad \partial_y^2 \prod_e f_e(y, v_e) \zeta_{E_0}^{r_{E_0}} &= \left(\sum_{\substack{e \neq e' \\ e, e' \notin E_0}} \frac{v_e v_{e'}}{4y^2} \partial_{v_e} \partial_{v_{e'}} + \sum_{e \notin E_0} \frac{2v_e^2 \partial_{v_e}^2 + 3v_e \partial_{v_e}}{2y^2} \right) \prod_{\tilde{e}} f_{\tilde{e}}(y, v_{\tilde{e}}) \zeta_{E_0}^{r_{E_0}}. \end{aligned}$$

In addition, using the equation $\Delta_e^H f_e(y, v_e) = 0$, we infer

$$(3.11) \quad \partial_{v_e}^2 f_e(y, v_e) = 8\pi L_e \frac{v_e}{y} \partial_{v_e} f_e(y, v_e).$$

We start by expanding (3.10):

$$\begin{aligned} & \mathcal{C}_{k,L}^J a(y) q^n \prod_e f_e(y, v_e) \zeta_{E_0}^{r_{E_0}} \\ &= -2(\Delta_{k-N/2} a(y) q^n) \prod_e f_e(y, v_e) \zeta_{E_0}^{r_{E_0}} \\ (3.12) \quad & - 8y^2 (\partial_\tau a(y) q^n) \partial_\tau \prod_e f_e(y, v_e) \zeta_{E_0}^{r_{E_0}} \\ (3.13) \quad & - 8y^2 (\partial_{\bar{\tau}} a(y) q^n) \partial_{\bar{\tau}} \prod_e f_e(y, v_e) \zeta_{E_0}^{r_{E_0}} \\ (3.14) \quad & + 2y^2 \left((\partial_\tau + \partial_{\bar{\tau}}) a(y) q^n \right) L_{\text{nd}}^{-1} \left[\frac{\pm i}{2} \partial_v \right] \prod_e f_e(y, v_e) \zeta_{E_0}^{r_{E_0}} \\ (3.15) \quad & - 8y (\partial_\tau a(y) q^n) (\pi_{\text{nd}} v)^T \partial_{\bar{z}} \prod_e f_e(y, v_e) \zeta_{E_0}^{r_{E_0}} \\ (3.16) \quad & + a(y) q^n \mathcal{C}_{k,L}^J \prod_e f_e(y, v_e) \zeta_{E_0}^{r_{E_0}}. \end{aligned}$$

In order to prove the proposition, it suffices to show that the sum of the last five terms vanishes. Since (3.12) to (3.16) involve either ∂_τ , $\partial_{\bar{\tau}}$, or L_{nd}^{-1} , we only need consider f_e for $L[e] \neq 0$.

Applying ∂_z and $\partial_{\bar{z}}$ to the product of all f_e is the same as applying $\pm \partial_y / 2$. From this, it is not hard to see that the sum of (3.12) and (3.13) equals

$$2y(a'(y) - 2\pi n a(y)) q^n \sum_{e \notin E_0} v_e \partial_e \prod_{\tilde{e}} f_{\tilde{e}}(y, v_{\tilde{e}}) \zeta_{E_0}^{r_{E_0}}.$$

By Theorem 3.5, the e for which f_e is not constant are orthogonal to all other $e' \in E$. This allows us to expand $L^{-2}[\partial_v] \prod_e f_e(y, v_e)$. Employing (3.11), we see that (3.14) equals

$$-2y 2\pi n a(y) q^n \sum_{e \notin E_0} v_e \partial_e \prod_{\tilde{e}} f_{\tilde{e}}(y, v_{\tilde{e}}) \zeta_{E_0}^{r_{E_0}}.$$

The fifth part, (3.15), can be expanded in a straight forward way:

$$-4y \left(\frac{1}{2} a'(y) - 2\pi n a(y) \right) q^n \sum_e v_e \partial_e \prod_{\tilde{e}} f_{\tilde{e}}(y, v_{\tilde{e}}).$$

Combining the parts that we have expanded so far, we see that (3.12) + (3.13) + (3.14) + (3.15) = 0. To prove the proposition we are, thus, reduced to showing that (3.16) vanishes.

We compute $\mathcal{C}_{k,L}^J \prod_e f_e(y, v_e)$. Considering expression (2.2) for $\mathcal{C}_{k,L}^J$, we see that all terms involving ∂_u do not contribute, because $r_e = 0$ in case $L[e] \neq 0$. The second term does not contribute either, because ∂_τ and $\partial_{\bar{\tau}}$ cancel each other if $L[e] \neq 0$, and they annihilate f_e , otherwise. Neither does the order four term contribute: For $L[e] = 0$, we find that $\zeta_e^{r_e}$ is annihilated by either ∂_{z_e} or $\partial_{\bar{z}_e}$. As above we can expand the other terms easily, when using the fact that the e for which f_e is not constant are orthogonal to all other. We are left with

the following expression:

$$\begin{aligned} & \mathcal{C}_{k,L}^J \prod_{e \notin E_0} f_e(y, v_e) \\ &= \left(-2y^2 \partial_y^2 - 2\left(k - \frac{N}{2}\right)y \partial_y - 2y \partial_y \sum_{e \notin E_0} v_e \partial_e \right. \\ & \quad \left. - \frac{1}{2} \sum_{e, e' \notin E_0} v_e v_{e'} \partial_{v_e} \partial_{v_{e'}} - \left(k - \frac{N}{2} - \frac{1}{2}\right) \sum_{e \notin E_0} v_e \partial_{v_e} \right) \prod_{\tilde{e} \notin E_0} f_{\tilde{e}}(y, v_{\tilde{e}}) \end{aligned}$$

Plugging in (3.9) and (3.10), shows

$$\mathcal{C}_{k,L}^J \prod_e f_e(y, v_e) \zeta_{E_0}^{r_{E_0}} = \prod_{e \in E_0} f_e(y, v_e) \zeta_{E_0}^{r_{E_0}} \cdot \mathcal{C}_{k,L}^J \prod_{e \notin E_0} f_e(y, v_e) = 0.$$

This completes the proof. \square

4. INDEFINITE THETA SERIES

In this section, we generalize the results of [Zwe02, Chapter 2]. Zwegers defined indefinite theta series for lattices of signature $(r-1, 1)$, $r \geq 2$. These theta series are H-harmonic, as can be deduced from Proposition 3.6. Zwegers also remarked that his construction generalized to arbitrary, non-degenerate lattices. In the previous section, though, we have seen that we may only expect real analytic contributions from the negative definite part of L . Further, these contributions must come from orthogonal parts of L . This does not put restriction on L , but on E . Indeed, for every suitable pair of frames (E, E') we will define an indefinite theta series. We will make a connection with Zwegers's construction in Example 4.5.

Throughout this section, we assume that L is non-degenerate.

Definition 4.1. *Let E be a partial frame of length $\mathrm{d}V_-$ such that $E^\perp / (E^\perp)_0$ is positive definite or the lattice of rank 0. Let E' be another such partial frame, and assume that the $\mathrm{span}(e_i, e'_i)$ are lattices of signature $(1, 1)$ that are mutually orthogonal for all $1 \leq i \leq \dim V_-$.*

Then (E, E') is called a compatible pair of partial frames of L .

Compatible pairs of partial frames will play an essential role, when constructing H-harmonic theta series. However, they do not always exist.

Proposition 4.2. *Suppose that (E, E') is a compatible pair of partial frames of L . Then $\mathrm{d}V_+ \geq \mathrm{d}V_-$.*

Proof. By definition, e_i and e'_i span a lattice of signature $(1, 1)$ for all $1 \leq i \leq \mathrm{d}V_-$. Since E and E' are both orthogonal the proposition follows by taking successive orthogonal complements in L . Indeed, $\mathrm{span}(e_1, e'_1)^\perp \subset L$ is a lattice of signature $(\mathrm{d}V_+ - 1, \mathrm{d}V_- - 1)$ which is equipped with the compatible pair of partial frames $(E \setminus \{e_1\}, E' \setminus \{e'_1\})$. Now, the proposition follows by induction. \square

In order to define indefinite theta series, set

$$(4.1) \quad \rho^e(\tau, z; \nu) = \begin{cases} \mathrm{sgn}(\langle e, \nu \rangle_L), & \text{if } L[e] = 0; \\ H^{\mathrm{H}[e]}(y, v; \nu), & \text{if } L[e] < 0 \end{cases}$$

for any vector $e \in L$.

Definition 4.3. *Let (E, E') be a compatible pair of partial frames of L . Define the (indefinite) theta series attached to E and E' as*

$$(4.2) \quad \theta_L^{E, E'}(\tau, z) = \sum_{\nu \in L} \left(\prod_{i=1}^{\mathrm{d}V_-} (\rho^{e_i} - \rho^{e'_i})(\tau, z; \nu) \right) e(L[\nu]\tau + 2\langle z, \nu \rangle_L).$$

Remark 4.4. If, up to scalar multiples, E and E' have a vector in common, we have $\theta_L^{E,E'} = 0$.

Example 4.5 (Zwegers's indefinite theta series for lattices of signature $(r-1, 1)$). In [Zwe02, Chapter 2], Zwegers analyzed indefinite theta series for Lorentzian lattices. We adopt his notation. Starting with V of signature $(r-1, 1)$, $r \geq 2$, choose isotropic or negative vectors $c_1, c_2 \in \mathbb{R}^{\text{rk}(L)}$. For any such choice, one obtains an indefinite theta series

$$\theta^{c_1, c_2}(\tau, z) = \sum_{\nu \in \mathbb{Z}^r} \rho^{c_1, c_2}(\nu; \tau) e(L[\nu]\tau + \langle \nu, z \rangle_L),$$

where, in this paper's notation, we have

$$\rho^{c_1, c_2}(\nu; \tau) = \rho^{c_1}(\tau, z; \nu) - \rho^{c_2}(\tau, z; \nu).$$

Setting $E = (c_1)$ and $E' = (c_2)$, we find that (E, E') is a compatible pair of partial frames as long as c_1 is not a multiple of c_2 . In the latter case, θ^{c_1, c_2} vanishes. In all other case, we find $\theta^{c_1, c_2} = \theta_L^{E, E'}$.

Set

$$D(E) := \{(\tau, z) \in \mathbb{H}_{1, N} : \langle e, v/y \rangle_L \notin \mathbb{Z} \text{ for all } e \in E_0\}.$$

Later on, we will prove

Proposition 4.6. If (E, E') is a compatible pair of partial frames, then series (4.2) converges absolutely for any $z \in D(E) \cap D(E')$.

Further, it can be analytically continued up to $\mathbb{H}_{1, \text{rk}(L)}$ except for non-moving singularities of almost meromorphic type.

Not all indefinite theta series are H-harmonic. The next Proposition clarifies this.

Proposition 4.7. Suppose that $\theta_L^{E, E'} \neq 0$ is H-harmonic. Then $E \cup E'$ contains at most dV_- vectors of negative length.

Furthermore, there is a compatible pair of partial frames (\tilde{E}, \tilde{E}') where \tilde{E}' contains only isotropic vectors such that $\theta_L^{E, E'} = \pm \theta_L^{\tilde{E}, \tilde{E}'}$.

Proof. Since the exponents of ζ are distinct for distinct $\nu \in L$, the proposition reduces to a purely algebraic questions. Because $\text{span}(e_i, e'_i)$ has signature $(1, 1)$, it follows that either e_i or e'_i must be isotropic. From this it is immediate that the number of vectors of negative length is at most dV_- .

To prove the second statement, consider $1 \leq i \leq dV_-$ such that $L[e_i] = 0$ and $L[e'_i] < 0$. It suffices to prove that $(E \setminus \{e_i\}) \cup \{e'_i\}$ and $(E' \setminus \{e'_i\}) \cup \{e_i\}$ form a compatible pair of partial frames. Fix $1 \leq j \leq dV_-, j \neq i$. We must show that e_j and e'_i , and e'_j and e_i are orthogonal. Since $\text{span}(e_i, e'_i)$ and $\text{span}(e_j, e'_j)$ are orthogonal, this follows immediately. \square

From this proposition we infer that the following definition covers all indefinite theta series for compatible pairs of frames that are H-harmonic.

Definition 4.8. An H-harmonic theta series is an indefinite theta series $\theta_L^{E, E'}$, for which E' contains only isotropic vectors. We call $\theta_L^{E, E'}$ a maximal H-harmonic theta series if E contains only vectors of negative length.

To prove the convergence of indefinite theta series, we will need to replace certain elements of E and E' by more suitable ones. The next lemma tells us that we can always replace isotropic vectors by negative ones.

Lemma 4.9. Let (E, E') be a compatible pair of partial frames of L . Then there is a vector \tilde{e} of negative length such that $(\{\tilde{e}\} \cup E \setminus \{e_i\}, E')$ and $(E, \{\tilde{e}\} \cup E' \setminus \{e'_i\})$ are compatible pairs of partial frames. In particular, we can choose \tilde{e} such that it has rational coordinates.

Proof. We analyze the Grassmanian of negative definite, real subspaces

$$H := G_- \left((E \cup E' \setminus \{e_i, e'_i\})^\perp \right).$$

We have $H \subseteq G(1, V)$, because for all $1 \leq j \leq \dim V_-$ we assume that $\text{span}(e_j, e'_j)$ is a lattice of signature $(1, 1)$. In particular, by Proposition 4.2, H is the Grassmanian of negative lines in an indefinite quadratic space. As such, it is a manifold of dimension greater than or equal to 1.

We must show that there is a point in $H \setminus (e_i^\perp \cup e'_i{}^\perp)$. Note that $e_i^\perp \subseteq G(1, V)$ is closed, and so is $e'_i{}^\perp$. Since $\overline{H} \setminus (e_i^\perp \cup e'_i{}^\perp) \subseteq G(1, V)$ contains one point, we find an open set in H that meets neither e_i^\perp nor $e'_i{}^\perp$. Any rational vector \tilde{e} spanning a space in this open set fulfills the requirements. \square

The next proof is essentially due to Zwegers [Zwe02]. A little care must be taken when splitting off $\text{span}(e_i, e'_i)$ and for this reason we give some details. It is also important that we prove that $\theta_L^{E, E'}$ has non-moving singularities, which was not mentioned in [Zwe02], even though it is immediate from the treatment given there.

Proof of Proposition 4.6. Using Lemma 4.9, we can successively write $\theta_L^{E, E'}$ as the sum of $\theta_L^{E, \tilde{E}'}$ and $\theta_L^{\tilde{E}, E'}$, where $\tilde{E} = \{\tilde{e}\} \cup E \setminus \{e_i\}$ and $\tilde{E}' = \{\tilde{e}\} \cup E' \setminus \{e'_i\}$. That is, in conjunction with Proposition 4.7, we can assume that $L[e_i] < 0$ for all $1 \leq i \leq dV_-$. Under this assumption we can proceed similarly to [Zwe02]. We write $\rho^{e_i} - \rho^{e'_i}$ as a sum or difference of terms

$$\begin{aligned} \text{sgn}(\langle e_i, \nu \rangle_L) \beta \left(-y \left(\nu - \frac{v}{y} \right)_{e_i}^2 / L_{e_i} \right), \quad \text{sgn}(\langle e'_i, \nu \rangle_L) \beta \left(-y \left(\nu - \frac{v}{y} \right)_{e'_i}^2 / L_{e'_i} \right), \quad \text{and} \\ \text{sgn}(\langle e_i, \nu \rangle_L) - \text{sgn}(\langle e'_i, \nu \rangle_L). \end{aligned}$$

It will be convenient to write z as $a + \tau b$ for $a, b \in \mathbb{R}^{\text{rk}(L)}$.

Consider disjoint sets $E^H \subseteq \{1, \dots, dV_-\}$ and $E'^H \subseteq \{1, \dots, dV_-\}$ such that $L[e_i] < 0$ for all $i \in E^H$ and $L[e'_i] < 0$ for $i \in E'^H$. Write E_{isotr} for $\{1, \dots, dV_-\} \setminus (E^H \cup E'^H)$. We are done, if we can show that for all such sets the series

$$\begin{aligned} \sum_{\nu \in \mathbb{Z}^r} \left(\prod_{i \in E^H} \text{sgn}(\langle e_i, \nu \rangle_L) \beta \left(-y \left(\nu - \frac{v}{y} \right)_{e_i}^2 / L_{e_i} \right) \prod_{i \in E'^H} \text{sgn}(\langle e'_i, \nu \rangle_L) \beta \left(-y \left(\nu - \frac{v}{y} \right)_{e'_i}^2 / L_{e'_i} \right) \right. \\ \left. \cdot \prod_{i \in E_{\text{isotr}}} \text{sgn}(\langle e_i, \nu \rangle_L) - \text{sgn}(\langle e'_i, \nu \rangle_L) \right) e(L[\nu]\tau + 2\langle z, \nu \rangle_L) \end{aligned}$$

converges locally absolutely.

Since e_i and e'_i span lattices of signature $(1, 1)$ for all i , and since E and E' are orthogonal partial frames, the estimates in the proof of [Zwe02, Proposition 2.4] apply word by word. We carry out some details for the convenience of the reader.

It suffices to deduce a crude estimate of the first and second factor. Upper bounds for each term are given by

$$\begin{aligned} \left| \left(\prod_{i \in E^H} \text{sgn}(\langle e_i, \nu \rangle_L) \beta \left(-y \left(\nu - \frac{v}{y} \right)_{e_i}^2 / L_{e_i} \right) \prod_{i \in E'^H} \text{sgn}(\langle e'_i, \nu \rangle_L) \beta \left(-y \left(\nu - \frac{v}{y} \right)_{e'_i}^2 / L_{e'_i} \right) \right. \right. \\ \left. \cdot \prod_{i \in E_{\text{isotr}}} \text{sgn}(\langle e_i, \nu \rangle_L) - \text{sgn}(\langle e'_i, \nu \rangle_L) \right) e(L[\nu]\tau + 2\langle z, \nu \rangle_L) \left| \right| \end{aligned}$$

$$\begin{aligned}
&\leq \prod_{i \in E^H} \exp \left(-2\pi y (L_{e_i}(\nu + a)_{e_i}^2 - \langle e_i, \nu + a \rangle_L^2 / (2L_{e_i})) \right) \\
&\quad \cdot \prod_{i \in E'^H} \exp \left(-2\pi y (L_{e'_i}(\nu + a)_{e'_i}^2 - \langle e'_i, \nu + a \rangle_L^2 / (2L_{e'_i})) \right) \\
&\quad \cdot \prod_{i \in E_{\text{isotr}}} |\text{sgn}(\langle e_i, \nu \rangle_L) - \text{sgn}(\langle e'_i, \nu \rangle_L)| \exp \left(-2\pi y (L_{e_i} \nu_{e_i}^2 + 2\langle z_{e_i}, \nu_{e_i} \rangle_L) \right).
\end{aligned}$$

We are reduced to bounding the third factor. Recall that $L[e_i] < 0$ for all $i \in E$, in particular, for all $i \in E_{\text{isotr}}$. By replacing e_i , $i \in E_{\text{isotr}}$ by a suitable multiple, we can assume that $e_i \in L$ (We drop the assumption $e_i^2 = 1$ for the time being. It will not be used till the end of the proof.) We decompose ν as $\mu + \sum_{i \in E_{\text{isotr}}} n_i e_i$, where $0 \leq \langle e_i, \mu \rangle_L \langle e_i, e'_i \rangle_L^{-1} < 1$. As in the proof of [Zwe02, Proposition 2.4], we can estimate the above product by

$$(4.3) \quad \frac{2}{1 - e(\langle e'_i, \mu \rangle_L \tau + \langle e'_i, b_e \rangle_L)}.$$

We have used the fact that $z \in D(E')$. Now, it is clear how to reduce the remaining sum to a finite sum of theta-like series for definite lattices as in [Zwe02].

We are left with proving that $\theta_L^{E, E'}$ has non moving singularities. Since (4.3) can be continued to all except finitely many $b \pmod{1}$, this follows from the fact that $\theta_L^{E, E'}$ is expressed as a finite sum of theta-like series. These singularities are of the right type, because (4.3) leads to holomorphic singularities. \square

Theorem 4.10. *Suppose that L is an even lattice. Then the $(\theta_L^{E, E'} | (0, \lambda))_{\lambda \in \text{disc}(L)}$ are the components of a vector valued Maaß-Jacobi form of weight $\text{rk}(L) / 2$, index L , and type ρ_L^J .*

Corollary 4.11. *Suppose that L is unimodular and E' contains only isotropic vectors. Then $\theta_L^{E, E'}$ is an $H[E]$ -harmonic Maaß-Jacobi form of weight $\text{rk}(L) / 2$ and index L .*

The next lemma will help us proving Theorem 4.10.

Lemma 4.12. *Let e be a continuous function $e : [0, \infty) \rightarrow \text{span}(e_i, e'_i)$ for some fixed i . Assume that $L[e(t)] < 0$ for $t > 0$, and $e(0) = e_i$. Set $E(t) = \{e(t)\} \cup (E \setminus \{e_i\})$. Then $\lim_{t \rightarrow 0} \theta_L^{E(t), E'} = \theta_L^{E, E'}$.*

Proof. The proof of [Zwe02, Proposition 2.7 (4)] applies almost word by word. The only crucial fact used there is that $\text{span}(e(t), e'_i)$ has signature $(1, 1)$ for all t . In our case this holds by Definition 4.1. Recalling that the spaces $\text{span}(e_i, e'_i)$ are orthogonal one to another for $0 \leq i \leq dV_-$, it is easy to see that it suffices to give a uniform bound for $\rho^{e(t)} - \rho^{e'_i}$. This bound can be obtained as in [Zwe02]. \square

Proof of Theorem 4.10. In the light of Lemma 4.12, the statement follows in a straight forward way, when combining the ideas in [Zwe02, Proposition 2.7 (7)] and those contained in the proof of Proposition 4.6. For this reason, we only sketch how to obtain the general case of [Zwe02, Lemma 2.8].

To calculate the relevant Fourier transform as in [Zwe02, Lemma 2.8], it suffices (we adopt the notation used there) to consider the derivative $\prod_{e \in E} \partial_{\alpha_e}$ in all directions of E . Splitting off the negative definite part allows us to use the usual Fourier transformation formula. In

order to adopt the last part of the proof of [Zwe02, Lemma 2.8], it suffices to notice that

$$\begin{aligned} & \int_{\mathbb{R}^{\text{rk}(L)}} e(L[a]\tau + \langle a, \alpha \rangle_L) \prod_{i=1}^{\text{d}V_-} (\rho^{e_i} - \rho^{e'_i})(\tau, a\tau; 0) da \\ & - \frac{i}{\sqrt{|L|}(-i\tau)^{\text{rk}(L)/2}} e(-L[\alpha]/\tau) \prod_{i=1}^{\text{d}V_-} (\rho^{e_i} - \rho^{e'_i})(-1/\tau, -\alpha/\tau; 0) \end{aligned}$$

is an odd function of α_e for all $e \in E$. \square

Remark 4.13. As in [Zwe02], one can show that $\theta_L^{E, E'}$ is analytic as a function on a suitable subset of $G_-(V) \times \mathcal{O}_{\text{d}V_-}^2$. This variety, however, does not contain any points for which $\theta_L^{E, E'}$ is H-harmonic.

5. SPECIALIZATIONS TO TORSION POINTS

In the first part of this section, we prove Theorem 5.2, from which Theorem 1.2 follows. The second part contains several examples of H-harmonic Maaß-Jacobi forms whose specializations to torsion points are of particular interest. Throughout the section, we assume that L is non-degenerate. We also assume, for our convenience, that E is a frame, that is, $\text{span}(E) = \mathbb{R}^{\text{rk}(L)}$.

We start by constructing a generalization of the $\hat{\mu}_{m, l}$ in [BRR12]. Recall the definition ($0 < m \in \mathbb{Z}$, $l \in \mathbb{Z}$):

$$\begin{aligned} \hat{\mu}_{m, l}(\tau, z) &:= \frac{(-1)^m}{\sqrt{m}} q^{\frac{-(l+m)^2}{4m}} \zeta^{-(l+m)} \\ &\cdot \left(\mu_m(\tau, 2mz + (l+m)\tau + \frac{1}{2}, \frac{1}{4m}) - \frac{i}{2} R(2m\tau, 2mz + (l+m)\tau - \frac{2m+1}{2}) \right). \end{aligned}$$

We have used

$$\begin{aligned} \mu_m(\tau, z_1, z_2) &:= \frac{e(z_1/2)}{\theta(\tau, z_2)^{2m}} \sum_{n \in \mathbb{Z}^{2m}} \frac{(-1)^{|n|} q^{\frac{1}{2}\|n\|^2 + \frac{1}{2}|n|} e(z_2|n|)}{1 - e(z_1)q^{|n|}}, \\ \theta(z; \tau) &:= \sum_{r \in \mathbb{Z} + \frac{1}{2}} (-1)^{r + \frac{1}{2}} q^{\frac{r^2}{2}} \zeta^r, \quad \text{and} \\ R(\tau, z) &:= \sum_{r \in \mathbb{Z} + \frac{1}{2}} \left(\text{sgn}(r) + \frac{1}{\sqrt{\pi}} H^{\text{H}[(1)]}(y, z; r) \right) (-1)^{r + \frac{1}{2}} q^{-\frac{r^2}{2}} \zeta^r. \end{aligned}$$

(drop the normalization for z_e) Assume that $\text{span}(E) = \mathbb{R}^{\text{rk}(L)}$ contains no isotropic vectors. Fix some $A \in M_{\text{rk}(L)}(\mathbb{Z})$ that leaves $\text{span}(E_+)$ and $\text{span}(E_-)$ invariant such that $\Lambda = A^T L A = \bigoplus_{e \in E} \Lambda_e$. Given $l \in \text{disc}(L)$, choose a representative in $L^\vee \subseteq \Lambda^\vee$. This way we can interpret l as an element of $\text{disc}(\Lambda)$. We write l_e for the corresponding components in $\text{disc}(\Lambda) = \bigoplus \text{disc}(\Lambda_e)$. Define

$$(5.1) \quad \hat{\mu}_{L, l}(\tau, z) := \sum_{\lambda \in L / \Lambda} \prod_{e \in E_+} \theta_{\Lambda_e, (l+\lambda)_e}(\tau, (A^{-1}z)_{A^{-1}e}) \prod_{e \in E_-} \hat{\mu}_{\Lambda_e, (l+\lambda)_e}(\tau, (A^{-1}z)_{A^{-1}e}).$$

The holomorphic part of $\hat{\mu}_{L, l}$ depends on A . Here, and throughout the section, we assume that the $\mu_{L, l}$ only depend on L and l . That is, for each L , we fix A once and for all.

Proposition 5.1. For any non-degenerate lattice L and any frame E that contains no isotropic vectors,

$$(\hat{\mu}_{L, l})_{l \in \text{disc}(L)}$$

is a vector valued $H[E]$ -harmonic Maaß-Jacobi form of weight $\mathrm{rk}(L)/2$, index L , and type ρ_L . Its image under $\xi^{H[E-]}$ is

$$\left(\sum_{\lambda \in L/\Lambda} \prod_{e \in E_+} \theta_{\Lambda_e, (l+\lambda)_e}(\tau, (A^{-1}z)_{A^{-1}e}) \prod_{e \in E_-} \overline{\theta_{\Lambda_e, (l+\lambda)_e}(\tau, (A^{-1}z)_{A^{-1}e})} \right)_{l \in \mathrm{disc}(\Lambda)}.$$

Furthermore, it is annihilated by ξ .

Proof. First assume that A is the identity matrix, so that we have $L = \Lambda$. Then it is clear that the function is a vector-valued Maaß-Jacobi form, since it is a tensor product of such forms. A straight forward computation of the Fourier expansion shows that it is H -harmonic and that it is annihilated by ξ . The image under $\xi^{H[E-]}$ can be either computed in the same way, or it can be deduced from the image of $\widehat{\mu}_{m,l}$ under ξ^H as in [BRR12].

In the general case, the correct transformation behavior under $\mathrm{SL}_2(\mathbb{Z}) \subseteq \Gamma_{\mathrm{rk}(L)}^J$ follows immediately from

$$\sum_{\lambda \in L/\Lambda} e(\langle l, \lambda \rangle_L) = \begin{cases} |L/\Lambda|, & \text{if } l \in L; \\ 0, & \text{otherwise.} \end{cases}$$

To show that the $\widehat{\mu}_{L,l}$ transform correctly under the Heisenberg part of $\Gamma_{\mathrm{rk}(L)}^J$, note that it is still $\mathbb{Z}^{\mathrm{rk}(L)}$ -periodic in z , since $l \in \mathrm{disc}(L)$. It is also invariant under the transformation $z \mapsto z + \lambda\tau$, because $\widehat{\mu}_{m,l} = \widehat{\mu}_m|_{(I_2, \frac{l}{2m}, 0)}$. That is, we have

$$\begin{aligned} & \sum_{\lambda \in L/\Lambda} \prod_{e \in E_+} \theta_{\Lambda_e, l_e}(\tau, (A^{-1}(z + \lambda\tau))_{A^{-1}e}) \prod_{e \in E_-} \widehat{\mu}_{\Lambda_e, l_e}(\tau, (A^{-1}(z + \lambda\tau))_{A^{-1}e}) \Big|_{k,L} (I_2, \lambda', \mu) \\ &= \sum_{\lambda \in L/\Lambda} \prod_{e \in E_+} \theta_{\Lambda_e, l_e}(\tau, (A^{-1}(z + (\lambda + \lambda')\tau))_{A^{-1}e}) \prod_{e \in E_-} \widehat{\mu}_{\Lambda_e, l_e}(\tau, (A^{-1}(z + (\lambda + \lambda')\tau))_{A^{-1}e}). \end{aligned}$$

From this, we see that $z \mapsto z + \lambda$ leaves $\widehat{\mu}_{L,l}$ invariant. This completes the proof. \square

In the statement of the next theorem, for the sake of notational convenience, we assume that $\mathrm{span}(E) = \mathbb{R}^{\mathrm{rk}(L)}$.

Theorem 5.2. *Let ϕ be an $H[E]$ -harmonic Maaß-Jacobi form of index L . Then there is the following decomposition of ϕ :*

$$(5.2) \quad \phi(\tau, z) = \sum_{E' \subset E_-} \phi_{E'}^{\mathrm{mero}}(\tau, z_{E_+ \cup E'}) \sum_{l \in \mathrm{disc}(L + \Lambda_{E'})} h_{E',l}(\tau) \widehat{\mu}_{L + \Lambda_{E'},l}(\tau, z).$$

Here, $\Lambda_{E'}$ are lattices supported on $\mathrm{span}(E_+ \cup E')$, and $\phi_{E'}^{\mathrm{mero}}$ are meromorphic Jacobi forms of index $L_{E_+ \cup E'} + \Lambda_{E'}$. The $h_{E',l}$ are components of a modular form in $M_k^1(\rho_{L + \Lambda_{E'}})$.

Remark 5.3. *Since the definition of $\widehat{\mu}_{L,l}$ is not canonical, the decomposition in Theorem 5.2 is also not canonical.*

The proof of Theorem 5.2 depends on several lemmas that we will state after deducing Theorem 1.2, which is a consequence of Theorem 5.2.

Proof of Theorem 1.2. The theorem follows immediately from Proposition 3.6, if $z(\tau)$ does not meet the divisors of ϕ^{mero} and $\widehat{\mu}_{L,l}$. In all other cases, for each E' and l , we choose a meromorphic Jacobi form $\psi_{E',l}$ according to Lemma 5.4, which we will state and prove afterward, that cancels all divisors that intersect $z(\tau)$. The assumptions of the lemma are met, since the principle part of $\widehat{\mu}_{L + \Lambda_{E'},l}$ occurs as the principle part of a meromorphic Jacobi form (see [BRR12] for details).

Consider the expression

$$\begin{aligned} \sum_{E' \subset E_-} \sum_{l \in \text{disc}(L + \Lambda_{E'})} \left(\phi_{E'}^{\text{mero}}(\tau, z_{E_+ \cup E'}) h_{E', l}(\tau) \widehat{\mu}_{L + \Lambda_{E'}, l}(\tau, z) - \psi_{E', l} \right) \\ + \sum_{E' \subset E_-} \sum_{l \in \text{disc}(L + \Lambda_{E'})} \psi_{E', l}. \end{aligned}$$

The sum in the first line is regular at $z(\tau)$ by choice of $\psi_{E', l}$. Specializing to $z(\tau)$, we obtain a product of harmonic weak Maaß forms as stated. Since, by assumption, ϕ has no singularities at $z(\tau)$, the sum in the second line cannot have singularities at $z(\tau)$, either. It yields a weakly holomorphic modular form when specializing to $z(\tau)$. This proves the theorem. \square

Lemma 5.4. *Let \mathcal{D} be a set of finitely many divisors in $\mathbb{H}_{1, \text{rk}(L)}$ of the form $\lambda(z) = \alpha\tau + \beta$ for some $\lambda \in (\mathbb{C}^{\text{rk}(L)})^\vee$, $\alpha, \beta \in \mathbb{Q}$. For each $D \in \mathcal{D}$ fix a principle part P_D that occurs as the principle part of a weight k meromorphic Jacobi. Fix a divisor D_s that is also of the form $\lambda(z) = \alpha\tau + \beta$.*

Then there is a meromorphic Jacobi form ψ of weight k that has singularities along all $D \in \mathcal{D}$ with principle part P_D , and the other singularities of ψ do not meet D_s .

Proof. This is easy to prove by first constructing suitable Jacobi forms ϕ of scalar index using theta series, and then considering sums of appropriate $\phi(\tau, \lambda(z))$. \square

We will need the next lemmas to prove Theorem 5.2.

Lemma 5.5. *Let $\phi \neq 0$ be an E -skew Jacobi form for a frame E satisfying $\text{span}(E) = \mathbb{R}^{\text{rk}(L)}$. Then ϕ has no singularities.*

Proof. By definition, singularities of ϕ are non-moving. Assume that ϕ has singularities, and let \tilde{z} be a singular point such that ϕ has no singularities in a neighborhood U of \tilde{z} outside a hyperplane given by an equation $\lambda(z) = \alpha\tau + \beta$ for $\lambda \in (\mathbb{C}^{\text{rk}(L)})^\vee$ and $\alpha, \beta \in \mathbb{Q}^{\text{rk}(L)}$. After changing coordinates, we can assume that $\lambda(z) = z_1$ and $\tilde{z} = (\tilde{z}_1, 0, \dots)$. The Laurent expansion of ϕ has the form

$$\phi(\tau, z) = \sum_{n \geq N} \psi_n(\tau, z) \overline{(z_1 - \alpha\tau - \beta)}^n$$

for some $N \in \mathbb{Z}$. We may assume that N is minimal with the property that $\psi_N \neq 0$.

Since L is non-degenerate by this section's assumptions, the highest derivative with respect to $\overline{z_1}$ that occurs in $\overline{\mathbb{L}_L}$ is a non-zero multiple of $\partial_{\overline{z_1}}^2$. The skew-meromorphic Jacobi form $\phi(\tau, z)$ vanishes under $\overline{\mathbb{L}_L}$. Applying $\overline{\mathbb{L}_L}$ to the above Laurent expansion, we find that

$$\psi_N(\tau, z) N(N-1) \overline{(z_1 - \alpha\tau - \beta)}^{N-2} = 0.$$

This shows that $N = 0$ or $N = 1$. Consequently, ϕ has no singularity at \tilde{z} . This contradicts our choice of \tilde{z} . \square

Lemma 5.6. *Let ϕ be an E' -skew Jacobi form for some $E' \subseteq E$. Assume that $\text{span } E = \mathbb{R}^{\text{rk}(L)}$. If $L = L_{E'} \oplus L_{E \setminus E'}$, then there is a lattice $\Lambda = \bigoplus_{e \in E \setminus E'} \Lambda_e$ and a Jacobi form ψ of index Λ such that $\psi(\tau, z_{E \setminus E'}) \phi(\tau, z)$ has no singularities.*

Proof. We show that any singularity of ϕ is supported at $\lambda(z_{E \setminus E'}) = \alpha\tau + \beta$ for some linear form $\lambda \in \text{span}(E \setminus E')^\vee$ and $\alpha, \beta \in \mathbb{Q}$. Then the claim follows, because all such divisors can be realized as the divisors of Jacobi forms whose index is supported on $\text{span}(E \setminus E')$ (see Lemma 5.4).

Since ϕ has non-moving singularities, there is an open dense subset $V \subseteq \text{span}(E \setminus E')$ such that for all $\lambda \in V$ the specialization $\psi := (\phi|_{(I_2, \lambda, 0)})(\tau, z_{E'})$ is well-defined as a function

with singularities. It is an E' -skew Jacobi form of index $L_{E'}$, as can be seen from the local Fourier expansion of ϕ . Since ψ is the restriction of ϕ to a hyperplane, it has non-moving singularities. By Lemma 5.5, it has no singularities and the claim follows immediately. \square

Lemma 5.7. *Let $\phi \neq 0$ be an $H[E]$ -harmonic Maaß-Jacobi form. Fix a partial frame $E' \subseteq E$.*

- (1) *Adopting the notation of Proposition 3.6, suppose that $c^{H[E']}(n, r) \neq 0$ for some n, r . Then $L = L_{E'} \oplus L_{E \setminus E'}$.*
- (2) *Suppose that $L = L_{E'} \oplus L_{E \setminus E'}$ and $\xi^{H[E']} \phi \neq 0$, while $\xi^{H[E'']} \phi = 0$ for all $E' \subset E'' \subseteq E$ for which $L = L_{E''} \oplus L_{E \setminus E''}$. Then there is a lattice $\Lambda = \bigoplus_{e \in E \setminus E'} \Lambda_e$, a meromorphic Jacobi form ϕ^{mero} of index Λ , and $(h_l)_{l \in \text{disc}(L+\Lambda)} \in M_{k - \frac{\text{rk}(L)}{2}}^1(\rho_{L+\Lambda})$ such that*

$$(5.3) \quad \phi(\tau, z) - \phi^{\text{mero}}(\tau, z_{E \setminus E'}) \sum_{l \in \text{disc}(L+\Lambda)} h_{E', l}(\tau) \mu_{L+\Lambda, l}(\tau, z)$$

vanishes under $\xi^{H[E']}$.

The image of ϕ and (5.3) under $\xi^{H[E'']}$ for all $E'' \not\subseteq E'$ is the same.

Proof. We first prove (2) assuming that (1) holds for all $E'' \subset E$ that contain E' . Second, we prove (1) assuming that (2) holds for all $E'' \subset E$ that contain E' . Then the lemma follows by induction on the order $(\{E' \subset E\}, \subset)$.

In order to prove (2), write $\tilde{\phi}$ for $\xi^{H[E']} \phi$. By assumptions and (1), this is an E' -skew Jacobi form. Applying Lemma 5.6 to $\tilde{\phi}$, we obtain Λ and ψ such that $\psi \tilde{\phi}$ has no singularities. Any E' -skew Jacobi form without singularities has a theta decomposition. We find

$$(5.4) \quad \psi(\tau, z_{E \setminus E'}) \tilde{\phi}(\tau, z) = \sum_{l \in \text{disc}(L+\Lambda)} h_l(\tau) \prod_{e \in E \setminus E'} \theta_{L_e + \Lambda_e, l_e}(\tau, z_e) \overline{\prod_{e \in E'} \theta_{L_e, l_e}(\tau, z_e)}.$$

By Proposition 5.1, $\xi^{H[E']}$ maps $\sum h_l \mu_{L+\Lambda, l}$ to (5.4). Setting $\phi^{\text{mero}} = \psi^{-1}$, we see that (5.3) vanishes under $\xi^{H[E']}$.

Now, we prove (1) by assuming that there is some $\phi \neq 0$ and some E' violating the statement. We can assume that E' is maximal with this property. By subtracting suitable Jacobi forms arising from (2), we can further assume that $c^{H[E'']}(n, r) = 0$ for all n, r and $E' \subset E''$.

Inspecting the Fourier expansion of ϕ , we see that

$$y^{-\#E'/2} \exp(-4\pi L_{E'}[v_{E'}]/y) \left(\prod_{e \in E'} Y_{-,e}^{k,L} \right) (\phi)$$

is nonzero and antiholomorphic with respect to $z_{E'}$. Since the differential operators that we have applied are covariant, we see that ϕ is invariant under a certain slash action. The cocycle of this slash action evaluated at $((\tau, z), (I_2, \lambda, 0))$ equals

$$\exp(2\pi i(\langle z, \lambda \rangle_L + L[\lambda]\tau)) \exp(-2\pi \langle v, \lambda \rangle_{L_{E'}} - 4\pi y L_{E'}[\lambda]).$$

This is not antiholomorphic in $z_{E'}$ if $L \neq L_{E'} \oplus L_{E \setminus E'}$, proving statement (1). \square

Proof of Theorem 5.2. We prove the theorem by repeated application of Lemma 5.7.

Fixed an integer $N > 0$. Suppose that ϕ vanishes under all $\xi^{H[E']}$, where $E' \subseteq E$, $\#E' > N$ satisfies $L = L_{E'} \oplus L_{E \setminus E'}$. Given $E' \subseteq E$, $\#E' = N$ satisfying $L = L_{E'} \oplus L_{E \setminus E'}$, apply Lemma 5.7 in order to obtain some

$$\phi_{E'}^{\text{mero}}(\tau, z_{E \setminus E'}) \sum_l h_l(\tau) \hat{\mu}_{L+\Lambda_{E \setminus E'}}(\tau, z).$$

Subtracting this, we obtain a Maaß-Jacobi form that is annihilated by $\xi^{H[E']}$. Its image under all $\xi^{H[E'']}$, $\#E'' = N$ is the same as before.

Starting with $N = \#E$, the theorem can now be proved by induction. \square

5.1. Examples of specializations to torsion points. In order to illustrate the connection of H-harmonic theta series to modular forms for $\mathrm{SL}_2(\mathbb{R})$, we give three examples.

Example 5.8 (Mock theta functions). In [Zwe02, Section 4.3], Zwegers used the following lattice to analyze Ramanujan's seventh order mock modular forms: Set $L = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$. We write b_1, b_2 for the standard basis vectors of \mathbb{R}^2 . Set $E = (-3b_1 + 4b_2)$ and $E' = (4b_1 - 3b_2)$. Writing I_2 for the identity in $\mathrm{SL}_2(\mathbb{Z})$, and s for $b_1 + b_2$, we see that

$$\left(\theta^{E, E'} \Big|_{k, L} \left(I_2, \frac{s}{14}, \frac{s}{14} \right) \right)_{z=0}$$

is the modular completion of $\eta(\tau)\mathcal{F}_0(\tau)$. Here, η is the Dedekind η -function, and \mathcal{F}_0 is one of the mock theta functions defined in [Ram00, page 355].

Any auxiliary isotropic vector in L , can be used to write $\eta(\tau)\mathcal{F}_0(\tau)$ as the sum of the specialization of two H-harmonic theta series. We do not make this explicit, since we think the next example illustrates this principle sufficiently well.

Example 5.9 (Products of harmonic weak Maaß forms). We use the ideas behind the previous example to find products of harmonic weak Maaß forms as specializations of theta series. Keep the same L as in Example 5.8, and set $L_2 = L \oplus L$. Let b_1, \dots, b_4 denote the standard basis vectors for \mathbb{R}^4 , and choose an isotropic vector $l \in L$. Let l_1 and l_2 denote its images under the two obvious embeddings of L into L_2 . Set

$$\begin{aligned} E_1 &= (-3b_1 + 4b_2, -3b_3 + 4b_4), & E'_1 &= (l_1, l_2); \\ E_2 &= (l_1, -3b_3 + 4b_4), & E'_2 &= (4b_1 - 3b_2, l_2); \\ E_3 &= (-3b_1 + 4b_2, l_2), & E'_3 &= (l_1, 4b_3 - 3b_4); \\ E_4 &= (l_1, l_2), & E'_4 &= (4b_1 - 3b_2, 4b_3 - 3b_4). \end{aligned}$$

Then (setting $s = b_1 + b_2 + b_3 + b_4$)

$$\left(\left(\theta^{E_1, E'_1} + \theta^{E_2, E'_2} + \theta^{E_3, E'_3} + \theta^{E_4, E'_4} \right) \Big|_{k, L_2} \left(I_2, \frac{s}{14}, \frac{s}{14} \right) \right)_{z=0}$$

is the modular completion of $\eta(\tau)^2\mathcal{F}_0(\tau)^2$. All the theta series which occur are H-harmonic, but the sum is not.

Remark 5.10. The preceding examples illustrates that the right analog of $\mathcal{MJ}_{k, m}^{\delta, H}$, that was studied in [BRR12], is $\sum_E \mathcal{MJ}_{k, L}^{\delta, H[E]}$, where E runs through all frames of L . This space, however, is too large, as Theorem 5.2 shows.

Example 5.11 (Products of holomorphic indefinite theta series). This example illustrates products of special holomorphic modular forms, that can be constructed as a corner case of 4.2. In [GZ98], they played a decisive role, and it is useful to have Jacobi forms, which specialize to products of them. We do not claim any originality for this quiet obvious construction, but to the authors knowledge, the details cannot be found in the literature.

We pick up [Zwe02, Example 2.16]. Set $L = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, and let $L_n = L^{\oplus n}$ be the n -fold copy of L . Then $E = (-b_{1+2i} + 2b_{2+2i})_{i=0, \dots, n-1}$ and $E' = (-2b_{1+2i} + b_{2+2i})_{i=0, \dots, n-1}$ form a pair of compatible partial frames of L_n . All vectors in E and E' are isotropic. Hence the associated indefinite theta series is holomorphic.

The following explicit expression follows immediately from [Zwe02, Equality (2.29)] (set $s = \sum_i b_i$):

$$\begin{aligned} & \left(\theta^{E,E'} \Big|_{k,L_n} (I_2, s/6, s/6) \right)_{z=0} \\ &= 2^n e(n/6) q^{\frac{n}{12}} \left(\left(\sum_{n,m \geq 0} - \sum_{n,m \leq 0} \right) (-1)^{n+m} q^{(n^2+4nm+m^2+n+m)/2} \right)^n \end{aligned}$$

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